Questions

1. Suppose we have a 1D image and we take the *local difference* of intensities,

$$DI(x) = \frac{1}{2}(I(x+1) - I(x-1))$$

which give a discrete approximation to a partial derivative. Or suppose we take a *local average* operation:

$$BI(x) = \frac{1}{4}I(x+1) + \frac{1}{2}I(x) + \frac{1}{4}I(x-1)$$

Define each of these operations as a convolution.

2. (a) What is $D(x) * B(x)$?
   (b) What is $D(x) * D(x)$?
   (c) What is $B(x) * B(x)$?

3. Let

$$I(x) * h(x) \;=\; -3\; I(x+2) \;+\; 4\; I(x+1) \;+\; 2\; I(x-2)$$

What is the function $h(x)$?

4. Let

$$I(t) * f(t) \;=\; 2I(t) - 3I(t-1) + I(t-2).$$

What is $f(t)$?

5. Prove the associative law for 1-D convolution,

$$(f * g) * h \;=\; f * (g * h)$$

You may assume that each of the three functions are defined over all the integers, but only have non-zero values over a finite range.

6. This question is for those of you who know some basic probability theory.

Suppose we have two independent random variables X and Y which can take integer values. Let the probabilities of X and Y be $f(X)$ and $g(Y)$, respectively. For example, the probability that $X = 3$ is $f(3)$ and the probabilty that $Y = -2$ is $g(-2)$.

Express the probability $h(Z)$ for the random variable $Z = X + Y$ in terms of a convolution.

7. What is the result of convolving $f(x)$ with a shifted delta function

$$f(x) * \delta(x - x_0) = \quad ?$$

8. What is the result of convolving a 1D sine function with $h(x)$?

Hint: in the lecture notes, I showed the result of convolving a 1D cosine function with $h(x)$. Use a similar idea here.

Solutions

1. We have $D(x) * I(x)$ or $B(x) * I(x)$ where

$$D(x) = \begin{cases} \frac{1}{2}, & x = -1 \\ -\frac{1}{2}, & x = 1 \\ 0, & \text{otherwise} \end{cases}$$

$$B(x) = \begin{cases} \frac{1}{4}, & x = -1 \\ \frac{1}{2}, & x = 0 \\ \frac{1}{4}, & x = 1 \\ 0, & \text{otherwise} \end{cases}$$

2. (a) By the definition of convolution

$$D(x) * B(x) = \sum D(x - x')B(x')$$

So, for example, take $x = 2$:

$$(D * B)(2) = \sum_{x'} D(2 - x')B(x') = D(1)B(1) = -\frac{1}{2} \cdot \frac{1}{4}$$

but all other terms are 0. Take $x = 1$:

$$(D * B)(1) = \sum_{x'} D(1 - x')B(x') = D(1)B(0) = -\frac{1}{2} \cdot \frac{1}{2}$$

but all other terms are 0. Take $x = 0$:

$$(D * B)(0) = \sum_{x'} D(-x')B(x') = D(-1)B(1) + D(1)B(-1) = \frac{1}{2} \cdot \frac{1}{4} - \frac{1}{2}\frac{1}{4} = 0.$$

but all other terms are 0.

$$D(x) * B(x) = \begin{cases} \frac{1}{8}, & x = -2 \\ \frac{1}{4}, & x = -1, \\ -\frac{1}{4}, & x = 1, \\ -\frac{1}{8}, & x = 2 \\ 0, & \text{otherwise} \end{cases}$$

Verify in Matlab by running `conv( [.5 0 -.5], [.25, .5, .25] )`

(b)

$$D(x) * D(x) = \begin{cases} \frac{1}{4}, & x = 2, -2 \\ -\frac{1}{2}, & x = 0, \\ 0, & \text{otherwise} \end{cases}$$

To verify, in Matlab run `conv([.5 0 -.5], [.5 0 -.5])`

(c)

$$B(x) * B(x) = \begin{cases} \frac{1}{16}, & x = 2, -2 \\ \frac{1}{4}, & x = -1, 1 \\ \frac{3}{8}, & x = 0 \\ 0, & \text{otherwise} \end{cases}$$

To verify, in Matlab run `conv([.25 .5 .25], [.25 .5 .25])` Also, notice that the result sums to 1, just as $B(x)$ sums to 1.

3. The function $h(x)$ is

$$h(x) = \begin{cases} -3, & x = -2 \\ 4, & x = -1 \\ 2, & x = 2 \\ 0, & \text{otherwise} \end{cases}$$

4. $f(0) = 2, f(1) = -3, f(2) = 1$. Or

$$f(t) = 2\delta(t) - 3\delta(t-1) + \delta(t-2)$$

5. All summations in the following are over $\infty, \ldots, \infty$.

$$\begin{aligned} (f*g)(x)*h(x) &= \sum_{x'} (f*g)(x') \;\; h(x-x') \\ &= \sum_{x'} \sum_{x''} f(x'')g(x'-x'') \;\; h(x-x') \\ &= \sum_{x''} f(x'') \sum_{x'} g(x'-x'') \;\; h(x-x') \\ & \text{let } x' - x'' = v \\ &= \sum_{x''} f(x'') \sum_{v} g(v) \;\; h((x-x'')-v) \\ &= \sum_{x''} f(x'')(g*h)(x-x'') \\ &= (f*(g*h))\;(x) \end{aligned}$$

6. To calculate the probability that Z takes a particular value, say $Z = z$, we need to consider all ways in which X and Y can sum z. For any $X = x$, the value of Y must be $z - x$. But $X = x$ with probability $f(x)$ and $Y = z - x$ with probability $g(z - x)$. Since X and Y are assumed to be independent, the probability of the event $(X, Y) = (x, z - x)$ is $f(x)g(z - x)$. Thus, the probability for Z=z must be

$$h(z) = \sum_{x=-\infty}^{\infty} f(x)g(z-x).$$

Thus, $h(z) = f * g(z)$, i.e. the convolution of f with g.

7. Let $g(x) = \delta(x - x_0)$.

$$
\begin{aligned}
f(x) * g(x) &= \sum_u f(x-u) g(u) \\
&= \sum_u f(x-u)\delta(u - x_0) \\
&= f(x - x_0) \ .
\end{aligned}
$$

since the delta function has value 1 when $u = x_0$ and 0 everywhere else. Thus, convolving $f(x)$ with a shifted delta function just shifts $f(x)$ by the same amount as the delta function.

8.

$$
h(x) * \sin(\frac{2\pi}{N} \ k \ x) \ = \ \sum_{x'=0}^{N-1} h(x') \ \sin(k\frac{2\pi}{N}(x - x')) \qquad (*)
$$

Recalling the trigonometry identity

$$
\sin(\alpha + \beta) \ = \ \sin\alpha\sin\beta + \cos\alpha\cos\beta
$$

we can expand the sin() on the right side of (*),

$$
\sin(k\frac{2\pi}{N}(x - x')) \ = \sin(k\frac{2\pi}{N}x) \ \cos(k\frac{2\pi}{N}(-x')) \ + \sin(k\frac{2\pi}{N}(-x')) \ \cos(k\frac{2\pi}{N}x)
$$

and so the right hand side of $(*)$ is just a sum of sine and cosine functions with variable x and constant frequency k, namely

$$
h(x) * \sin(k \ \frac{2\pi}{N} \ x) \ = \ a\cos(\frac{2\pi}{N} \ kx) + b\sin(\frac{2\pi}{N} \ kx)
$$

where

$$
a \ = \ \sum_{x'=0}^{N-1} h(x') \ \sin(k\frac{2\pi}{N}(-x'))
$$

$$
b = \sum_{x'=0}^{N-1} h(x') \ \cos(k\frac{2\pi}{N}(-x'))
$$

which are just the inner products of the N dimensional vectors $h(\cdot)$ with a cosine or sine of frequency k, respectively.